Transitive and quasitransitive actions of affine groups preserving a generalized Lorentz-structure

FRITZ GRUNEWALD

Mathematisches Institut
der Universität
Wegelerstraße 10
5300 Bonn 1

GREGORI MARGULIS (*)

Moscow, u. Ermolovoi 19,
Institute for Problems
in Information Transmission

Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *Let A^n be the n-dimensional affine space over $\mathbb{R}$ and $\mathbf{A}(n)$ its groups of motions. The map $\lambda : \mathbf{A}(n) \to GL_n(\mathbb{R})$ associates to an affine motion its linear part. In the first part of the paper we prove that any subgroup $\Gamma \leqslant \mathbf{A}(n)$ which acts discontinuously with compact quotient on A^n and which has the property that $\lambda(\Gamma)$ is contained in a Lie group of rank $\leqslant 1$ is polycyclic by finite. The second part of the paper classifies such groups which satisfy $\lambda(\Gamma) \leqslant O(n-1, 1)$ up to commensurability.*

1. INTRODUCTION

Let V be a real vectorspace of finite dimension n. We write $\mathbf{a}(V)$ for the affine space associated to V and $\mathrm{Aff}(V)$ for the group of affine motions on $\mathbf{a}(V)$. Choosing a basis in V we may make the following identifications:

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$$\mathbf{a}(\mathbb{R}^n) = \left\{ \begin{pmatrix} v^t \\ 1 \end{pmatrix} \mid v \in \mathbb{R}^n \right\}$$

$$\mathrm{Aff}(\mathbb{R}^n) = \left\{ \begin{pmatrix} g & w^t \\ 0 & 1 \end{pmatrix} \mid g \in GL_n(\mathbb{R}),\ w \in \mathbb{R}^n \right\}$$

The action of $\mathrm{Aff}(\mathbb{R}^n)$ on $\mathbf{a}(\mathbb{R}^n)$ is then given by the usual matrix product. A^t stands for the transpose of a matrix A. A subgroup $\Gamma \leqslant \mathrm{Aff}(V)$ is said to act *discontinuously* on $\mathbf{a}(V)$ if for every compact set $K \leqslant \mathbf{a}(V)$ the set

$$\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$$

is finite. A group which acts discontinuously is discrete in $\mathrm{Aff}(V)$, but the converse does not hold in general.

The group $\Gamma \leqslant \mathrm{Aff}(V)$ is said to act *quasitransitively* on $\mathbf{a}(V)$, if there is a compact set $K \leqslant \mathbf{a}(V)$ so that

$$\mathbf{a}(V) = \bigcup_{\gamma \in \Gamma} \gamma \cdot K.$$

The simplest examples of groups that act discontinuously and quasitransitively on $\mathbf{a}(V)$ are

$$\left\{ \begin{pmatrix} 1 & v^t \\ 0 & 1 \end{pmatrix} \mid v \in \Omega \right\}$$

where $\Omega \leqslant \mathbb{R}^n$ is a full lattice. It is easy to see that a group $\Gamma \leqslant \mathrm{Aff}(V)$ that acts discontinuously and quasitransitively on $\mathbf{a}(V)$ is finitely generated. The following is a long standing problem:

CONJECTURE 1.1. *Let V be a finite dimensional real vectorspace and $\Gamma \leqslant \mathrm{Aff}(V)$ a subgroup which acts discontinuously and quasitransitively on $\mathbf{a}(V)$. Then Γ is virtually polycyclic.*

If $\mathscr{P}$ is a property of groups, then the group G is called *virtually* $\mathscr{P}$ if G has a subgroup of finite index satisfying $\mathscr{P}$. A group G is called *polycyclic* if it has a series of subgroups:

$$\langle 1 \rangle = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$$

so that G_{i+1}/G_i is cyclic for all $0 \leqslant i \leqslant n-1$.

Originally conjecture 1.1 was posed even without the assumption that Γ acts quasitransitively [18]. In [17] the second author gave a counterexample to this stronger conjecture. Conjecture 1.1 has been solved affirmatively in case the dimension of V is 1, 2, 3, [9].

Let $\Gamma \leqslant \mathrm{Aff}(V)$ be a subgroup that acts discontinuously, quasitransitively and without fixed points on $\mathbf{a}(V)$. The latter means $\gamma P = P$ for $\gamma \in \Gamma$ and $P \in \mathbf{a}(V)$ implies $\gamma = 1$. In this case the set of orbits

$$\Gamma \backslash \mathbf{a}(V)$$

inherits from $\mathbf{a}(V)$ the structure of a complete, affinely flat compact manifold. It is easy to see that every such manifold arises as such a quotient. By a theorem of Selberg every finitely generated linear group contains a torsionfree subgroup of finite index. It follows that conjecture 1.1 can be given the equivalent form:

CONJECTURE 1.1′: *Let M be a complete, affinely flat compact manifold. Then its fundamental group $\pi_1(M)$ is virtually polycyclic.*

Given a finite dimensional vectorspace V we denote by

$$\lambda : \mathrm{Aff}(V) \to GL(V)$$

the canonical homomorphism which assign to an affine motion its linear part. The kernel of λ consists of the pure translations. In coordinates λ is given by:

$$\lambda : \begin{pmatrix} g & w^t \\ 0 & 1 \end{pmatrix} \to g.$$

For a subgroup $\Gamma \leqslant \mathrm{Aff}(V)$ we introduce its group of translations:

$$T_\Gamma = \Gamma \cap \left\{ \begin{pmatrix} 1 & w^t \\ 0 & 1 \end{pmatrix} \mid w \in V \right\},\ V_\Gamma = \left\{ v \in V \mid \begin{pmatrix} 1 & v^t \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}.$$

DEFINITION 1.2. Let V be a finite dimensional real vectorspace and $G \leqslant GL(V)$ a closed subgroup with finitely many connected components. A group $\Gamma \leqslant \mathrm{Aff}(V)$ is called *G-linear* if $\lambda(\Gamma) \leqslant G$.

The following are important special cases of the above concept. Let V have dimension n, take a basis, and write $\mathrm{O}(n)$, for the orthogonal group of the quadratic form

$$x_1^2 + \ldots + x_n^2$$

on the chosen basis. The $G = \mathbf{O}(n)$-linear groups are the groups of euclidean motions. The orthogonal group of the quadratic form

$$2x_0x_n + x_1^2 + \ldots + x_{n-1}^2$$

which is of signature $(n, 1)$ is denoted by $\mathbf{O}(n, 1)$. The $G = \mathbf{O}(n, 1)$-linear groups are the groups of Lorentz-motions.

Let $G \leqslant GL(V)$ be a closed subgroup. The real rank of G is denoted by

$$rk_{\mathbb{R}} G.$$

This is the maximal dimension of a subgroup of G isomorphic to an $\mathbb{R}$-split torus, that is $(\mathbb{R}^*)^d$. See Helgason [14] for the details in the theory of Lie groups. We prove here:

THEOREM 1.3. *Let V be a finite dimensional real vectorspace. Let $G \leqslant GL(V)$ be a closed subgroup with finitely many connected components which is reductive and satisfies $rk_{\mathbb{R}} G \leqq 1$. Then any G-linear subgroup $\Gamma \leqslant$ Aff (V) which acts discontinuously and quasitransitively on* $\mathbf{a}\,(V)$ *is virtually polycyclic.* ■

Theorem 1.3 has two predecessors. If $G = K \leqslant GL(V)$ is compact, that is $rk_{\mathbb{R}} G = 0$ then a G-linear group is conjugate to an $0(n)$-linear group. The theorem of Bieberbach [4] may be applied and proves that Γ is even virtually abelian.

If Γ is a group of Lorentz-motions, that is $\lambda(\Gamma) \leqslant \mathbf{O}(n, 1)$, $(n + 1 = \dim_{\mathbb{R}} V)$, then Goldman and Kamishima [11] have proved the above result. Our proof is similar to that of Goldman and Kamishima. We use induction on the dimension of V. We distinguish the cases that $\lambda(\Gamma)$ is discrete or not. In the first instance we use the same cohomological argument as [11]. In the second case our argument differs from that in [11]. We use a general description of the closed subgroups of the reductive groups of real rank 1.

We mention that a reductive real group of rank 1 is isogenous to one of the following types.

1) $\mathbb{R}^* \times K$,	K compact
2) $\mathbf{O}(m, 1) \times K$,	K compact, $m \geqslant 1$
3) $\mathbf{U}(m, 1) \times K$,	K compact, $m \geqslant 1$
4) $\mathbf{Sp}(m, 1) \times K$,	K compact, $m \geqslant 1$
5) $\mathbf{F}_4$ II $\times K$,	K compact.

We use here the terminology of Helgason [14]. The simbol $\times$ stands for almost direct product. If the group G mentioned in Theorem 1.3 is even semisimple then it is Zariski-closed. If not G is isogenous to $\mathbb{R}^* \times K$ where K is compact. In the latter case proposition 2.3 implies the statement of Theorem 1.3. Hence we can, without loss of generality, assume that G is an algebraic subgroup of $GL(V)$.

We proceed by investigating the following a bit vaguely stated problem.

PROBLEM 1.4. *What can one say about the isomorphism and conjugacy classes of virtually polycyclic groups that act discontinuously and quasitransitively on affine spaces?*

The following is easily, deduced from theorem 1.3. It shows that the structure of the virtually polycyclic groups arising in theorem 1.3 is quite restricted.

COROLLARY 1.5. *Let V be a finite dimensional real vector space. Let $G \leqslant GL(V)$ be a closed subgroup with finitely many connected components which is reductive and satisfies $rk_{\mathbb{R}} G \leqslant 1$. Then any G-linear subgroup $\Gamma \leqslant \mathrm{Aff}(V)$ which acts discontinuously and quasitransitively on $\mathbf{a}(V)$ has a series of subgroups*

$$\langle 1 \rangle \trianglelefteq \Gamma_0 \trianglelefteq \Gamma_1 \trianglelefteq \Gamma_2 \trianglelefteq \Gamma_3 \trianglelefteq \Gamma_4 = \Gamma$$

with

(i) Γ_0 *is abelian,*
(ii) Γ_1/Γ_0 *is nilpotent of class* $\leqslant 2$,
(iii) Γ_2/Γ_1 *is abelian,*
(iv) Γ_3/Γ_2 *is abelian,*
(v) Γ_4/Γ_3 *is finite.* ∎

Note that any subgroup of a virtually polycyclic group is finitely generated, so that we have a little more information about the subquotients occurring in Corollary 1.5.

The main tool in the finer investigation of problem 1.4 is a theorem proved by Fried and Goldman [9].

THEOREM 1.6. *Let V be a finite dimensional real vectorspace and $G \leqslant GL(V)$ a Zariski closed subgroup. Let $\Gamma \leqslant \mathrm{Aff}(V)$ be a G-linear virtually polycyclic group that acts discontinuously and quasitransitively on $\mathbf{a}(V)$. Then there is a subgroup $H \leqslant \mathrm{Aff}(V)$ which is G-linear with:*
(i) H *acts simply transitively on* $\mathbf{a}(V)$,
(ii) $H \cap \Gamma$ *has finite index in* Γ,
(iii) $H \cap \Gamma$ *is a lattice in* H, *i.e.* $H \cap \Gamma$ *is discrete and cocompact in* H. ∎

Theorem 1.6.splits Problem 1.4 virtually into two separate questions.

PROBLEM 1.4′. *Let V be a finite dimensional vectorspace $G \leqslant GL(V)$ a closed subgroup with finitely many connected components* *(i) Classify the subgroups* $H \leqslant \mathrm{Aff}(V)$ *which are G-linear and act simply transitively on* $\mathfrak{a}(V)$.

(ii) For each group which occurs in (i) classify the lattices in H.

Both parts are usually very difficult. A guiding line is given by the Bieberbach theorems (G = orthogonal group). Of course any group $H \leqslant \mathrm{Aff}(V)$ that acts simply transitively on $\mathfrak{a}(V)$ is a connected, simply connected Lie-subgroup of $\mathrm{Aff}(V)$. Auslander [1] has furthermore shown that H must be soluble. At the moment there is no soluble connected simply connected group which is known not to act simply transitively on some affine space. The first part of Problem 1.4′ can be put in stages of increasing difficulty. One might ask for a classification of the groups up to isomorphism or up to conjugacy in $\mathrm{Aff}(V)$.

Sections 3, 4, 5, 6 of our paper contain a treatment of Problem 1.4′ in case of Lorentz-motions that is $G = \mathbf{O}(n, 1)$. Our classifications contain the results of Auslander and Markus [3] ($n = 3$) and Fried [10] ($n = 4$) as special cases.

We shall describe our results now in more detail, For this let $W = \mathbb{R}^{n-1}$ be an $(n-1)$-dimensional vectorspace with basis. On the vectorspace of dimension $n+2$

$$\mathbb{R} \times W \times \mathbb{R}$$

We consider the quadratic form

$$q(x, v, y) = 2xy + vv^t$$

of signature $(n, 1)$. The group:

$$E(n, 1) = \left\{ \begin{pmatrix} g & w^t \\ 0 & 1 \end{pmatrix} \mid g \in \mathbf{O}(q, \mathbb{R}), w \in \mathbb{R} \times W \times \mathbb{R} \right\}$$

is called the group of affine Lorentz-motions. $\mathbf{O}(q, \mathbb{R})$ is the real orthogonal group of the quadratic form q.

First of all we describe our classification of the isomorphism types of unipotent subgroups $U \leqslant E(n, 1)$ which act simply transitively on affine space. For each dimension these groups fall into finitely many isomorphism types. We shall give presentations for the Lie algebras of the possible U.

DEFINITION 1.7. Let $n \geqslant 1$, $0 \leqslant k \leqslant (n-1)/2$ $i = 1, 2$ be integers. Let $\mathscr{V}$ be a real vectorspace of dimension $n+1$ with basis

$$\zeta, e_1, \ldots, e_k, f_1, \ldots, f_k, e_{2k+1}, \ldots, e_{n-1}, \tau.$$

On $\mathscr{V}$ we define the structure of Lie algebras $\mathscr{L}^i(n+1, k)$ by the multiplication tables in table 1. $\mathscr{L}^1(n+1, k)$ is defined for any

$$0 \leqslant k \leqslant \frac{n-1}{2}, \ \mathscr{L}^2(n-1, k) \text{ for } k \neq \frac{n-1}{2}.$$

We write $L^i(n+1, k)$ for the connected, simply connected Lie group with Lie algebra $\mathscr{L}^i(n+1.k)$

The Lie algebras $\mathscr{L}^i(n+1, k)$ are all nilpotent of nilpotency class $\leqslant 3$. They are mutually nonisomorphic.

THEOREM 1.8. *Let $n \geqslant 1$ be an integer and H a connected, simply connected nilpotent Lie-group of dimension $n+1$ Then the following two statements are equivalent*

(i) There is a subgroup $U \leqslant E(n, 1)$ which acts simply transitively on affine space and is isomorphic to H,

(ii) H is isomorphic to one of the $L^i(n+1, k)$.

This result is proved in section 3. We also give a set of representatives for the $E(n, 1)$ conjugacy classes of the simply transitive unipotent subgroups $H \leqslant E(n, 1)$.

The case of a general group $H \leqslant E(n, 1)$ acting simply transitively is treated in section 4. Such a group is necessarily connected, simply connected and soluble. In section 4 we give a detailed description of these groups which implies the following result.

THEOREM 1.9. *Let $n \geqslant 1$* be an integer and $H \leqslant E(n, 1)$ a subgroup that acts simply transitively on affine space. Then H is of one of the following types

(i) H is unipotent,

(ii) H is a split extension $\mathbb{R}^a \rtimes \mathbb{R}^b$ where $a + b = n + 1$ and $\mathbb{R}^b$ acts orthogonally on $\mathbb{R}^a$,

(iii) H is a split extension $\mathbb{R}^a \rtimes \mathbb{R}^b$ where $a + b = n + 1$ and $a \geqslant 2$, and $\mathbb{R}^b$ acts through a homomorphism

$$\mathbb{R}^b \to \mathbb{R}^* \times O(a-2) \to GL_a(\mathbb{R})$$

on $\mathbb{R}^a$, where $\mathbb{R}^*$ acts trivially up to one dimensional eigenspaces for the identical character and its inverse,

(iv) H is a sequence of split extensions

$$H = (H_1 \rtimes H_2) \rtimes H_3.$$

$H_1 = \mathbb{R}^a$, $H_2 = \mathbb{R}^b$, $H_3 = \mathbb{R}$ and $a + b + 1 = n + 1$. Here H_1 is normal in H and H acts orthogonally on H_1. The group H_3 acts trivially on the quotient $(H_1 \rtimes H_2)/H_1$,

(v) H is a sequence of split extensions

$$H = ((H_1 \times H_2) \rtimes H_3) \rtimes H_4,$$

$H_1 = \mathbb{R}^a$, *H_2 is an unipotent group with at most one dimensional commutator subgroup and of dimension b,* $H_3 = \mathbb{R}^c$, $H_4 = \mathbb{R}$ *and* $a + b + c + 1 = n + 1$. *Here* H_1 *and* $H_1 \times H_2$ *are normal in* H *and* H *acts orthogonally on* H_1 *and* H_2. H_3 *also normalizes* H_2 *and* H_4 *acts trivially on* $((H_1 \times H_2) \rtimes H_3)/H_1 \times H_2$.

In fact it is clear from section 5 that every group of type (ii), (iii), (iv) can be embedded into $E(n, 1)$ as a simply transitive group of affine motions. The type (v) has to be further restricted so that this is possible. We shall not discuss this here.

Next we want to explain our results on groups $\Gamma \leqslant E(n, 1)$ that act discontinuously and quasitransitively on affine space. Our aim is to describe up to finite index the isomorphism types of these groups. Remember that one of the Bieberbach theorems says that a quasitransitive and discontinuous group of euclidean motions on $\mathbb{R}^n$ contains a normal subgroup of finite index which is isomorphic to $\mathbf{Z}^n$.

THEOREM 1.10. *Let* $n \geqslant 1$ *be an integer, and* $\Gamma \leqslant E(n, 1)$ *a subgroup that acts discontinuously and quasitransitively on affine space. Then falls into one of the following two types:*

(i) Γ *is virtually nilpotent,*

(ii) Γ *is virtually (abelian by cyclic).*

Let $\mathscr{P}_0$, $\mathscr{P}_1$ be two properties of groups. A group G is called $\mathscr{P}_0$ by $\mathscr{P}_1$ if G. contains a normal subgroup H with property $\mathscr{P}_0$ so that the quotient G/H has property $\mathscr{P}_1$.

Theorem 1.10 can be considerably sharpened. The groups arising under (i) and (ii) can be further restricted. We first of all discuss the virtually nilpotent cases.

THEOREM 1.11. *Let* $n \geqslant 1$ *be an integer, and* $\Gamma \leqslant E(n, 1)$ *a virtually nilpotent subgroup that acts discontinuously and quasitransitively on affine space. Then* Γ *contains a subgroup* Γ_1 *of finite index with the following properties:*

(i) Γ_1 *is nilpotent of nilpotency class* $\leqslant 3$

(ii) Γ_1 *contains a normal subgroup* Γ_2 *which has cyclic or trivial commutator subgroup and so that* $\Gamma_1/\Gamma_2 \cong \mathbb{Z}$.

It is a simple matter to classify finitely generated nilpotent groups with cyclic commutator subgroups [13]. It so happens that not every cyclic extension of these groups occurs as a group $\Gamma \leqslant E(n, 1)$ that acts discontinuously and quasi-transitively on affine space. To clarify this point we introduce the following notion.

DEFINITION 1.12. Let Γ_0, Γ_1 be groups. They are said to be *abstractly commensurable* if there are subgroups $\theta_0 \leqslant \Gamma_0$ and $\theta_1 \leqslant \Gamma_1$ with:

(i) the indices $|\Gamma_i : \theta_i|$ are finite,

(ii) θ_0 and θ_1 are isomorphic.

For example Γ_0 and Γ_1 might lie in a common overgroup and intersect in a subgroup of finite index in both.

Let Γ_0, Γ_1 be two finitely generated torsionfree nilpotent groups. We write $M_{\mathbb{Q}}(\Gamma_i)$ for their rational Malcev-completions. The $M_{\mathbb{Q}}(\Gamma_i)$ are $\mathbb{Q}$ points of unipotent algebraic groups defined over $\mathbb{Q}$. Γ_0, Γ_1 are abstractly commensurable if and only if $M_{\mathbb{Q}}(\Gamma_0)$ and $M_{\mathbb{Q}}(\Gamma_1)$ are isomorphic as $\mathbb{Q}$-groups. This is the case if and only if their Lie algebras are isomorphic. For all of this see [13].

Let H be a connected, simply connected nilpotent Lie group. Then it is in general quite difficult to classify the abstract commensurability classes of lattices (i.e. cocompact discrete subgroups) in H. If H has a lattice then the Lie algebra $\mathscr{H}$ of H is defined over $\mathbb{Q}$. It can be proved [13] that the abstract commensurability classes of lattices in H correspond to the elements of the trivial fibre of the natural map of Galoiscohomology-groups

$$H^1(\mathbb{Q}, \mathrm{Aut}_{\mathbb{Q}}(\mathscr{H})) \to H^1(\mathbb{R}, \mathrm{Aut}_{\mathbb{Q}}(\mathscr{H})).$$

where $\mathrm{Aut}_{\mathbb{Q}}(\mathscr{H})$ is the automorphism group of $\mathscr{H}$ considered as an algebraic $\mathbb{Q}$-group.

We are here able to classify the abstract commensurability classes of the groups arising in Theorem 1.11 by equivalence classes of quadratic forms.

To do this we introduce the following groups.

DEFINITION 1.13. Let n, k be integers so that $0 \leqslant k \leqslant (n-1)/2$. For $m = (m_1, \ldots, m_k) \in (2 \cdot \mathbb{N})^k$ we define $\sqrt{m} = (\sqrt{m_1}, \ldots, \sqrt{m_k})$ and

$$L(\sqrt{m}) = \{(x_1\sqrt{m_1}, \ldots, x_2\sqrt{m_k}) \mid x_1, \ldots, x_k \in \mathbb{Z}\}.$$

We also define the positive definite quadratic forms

$$q_m = m_1 y_1^2 + \ldots + m_k y_z^2.$$

We furthermore put

$\Gamma_1(n+1, k, m) =$

$$=\left\{\begin{pmatrix} 1 & 0 & 0 & -x & -\frac{1}{2}xx^t & r \\ 0 & E & 0 & 0 & 0 & z^t \\ 0 & 0 & E & 0 & 0 & x^t \\ 0 & 0 & 0 & E & x^t & y^t \\ 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} r, s \in \mathbb{Z}; x, y \in L(\sqrt{m}); \\ z \in \mathbb{Z}^{n-1-2k} \end{matrix} \right\}$$

If $k < \frac{n-1}{2}$ we write $e_1 = (1, 0, \ldots . 0) \in \mathbb{Z}^{n-1-2k}$ and put

$\Gamma_2(n+1, k, m) =$

$$=\left\{\begin{pmatrix} 1 & -2se_1 & 0 & -x & -\frac{1}{2}(4s^2+xx^t) & r \\ 0 & E & 0 & 0 & 2se_1^t & z^t \\ 0 & 0 & E & 0 & 0 & x^t \\ 0 & 0 & 0 & E & x^t & y^t \\ 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} r, s \in \mathbb{Z}; x, y \in L(\sqrt{m}); \\ z \in \mathbb{Z}^{n-1-2k} \end{matrix} \right\}$$

The symbol E stands for the unit matrix of the appropriate dimension. The sets $\Gamma_i(n+1, k, m)$ are finitely generated torsionfree nilpotent groups that act discontinuously and quasitransitively on affine space. The groups $\Gamma_i(n+1, k, m)$ are all extensions where H_k is the discrete Heisenberg group of rank $2k+1$.

The groups $\Gamma_i\ (n+1, k, m)$ and $\Gamma_i\ (n'+1, k', m')$ are abstractly isomorphic if and only if $i = j$, $n = n'$, $k = k'$ and if the quadratic forms q_m and $q_{m'}$ are up to sign integrally equivalent.

THEOREM 1.14. *Let $n \geqslant 1$ be an integer*

(i) Let $\Gamma \leqslant E(n, 1)$ be a virtually nilpotent group that acts discontinuously and quasitransitively on affine space then Γ is abstractly commensurable to one of the groups $\Gamma_i(n+1, k, m)$.

(ii) The groups $\Gamma_i(n+1, k, m)$ and $\Gamma_j(n'+1, k', m)$ are abstractly commensurable if and only if the following hold $i = j$, $n = n'$, $k = k'$, and there is an

$\alpha \in \mathbb{Q}^*$ *so that the quadratic forms* q_m *and* $\alpha \cdot q_{m'}$ *are equivalent over* $\mathbb{Q}$.

The equivalent classes of nondegenerate quadratic forms over $\mathbb{Q}$ can be described by the Hasse-Minkowski theorem. The invariants of an equivalence are: dimension, determinant and the vector of Hasse symbols [8]. In Theorem 1.14 a slightly stronger equivalence relation amongst quadratic forms arises. We have described in Proposition 6.13 the modifications which have to be made to affine invariants for this relation.

We shall now give a description of the virtually abelian by cyclic groups which arise in Theorem 1.10, (ii).

DEFINITION 1.15. Let $n \geqslant 1$ be an integer and $A \in GL_n(\mathbb{Z})$ an invertible matrix. Let the cyclic group $\mathbb{Z}$ act on $\mathbb{Z}^n$ by

$$1 \cdot v = A \cdot v^t,$$

We write

$$\Gamma(n+1, A) = \mathbb{Z}^n \rtimes_A \mathbb{Z}.$$

Let $n \geqslant 3$. We call a matrix $A \in GL_n(\mathbb{Z})$ of *Lorentz type* if it is diagonalizable and its eigenvalues are

$$1, \lambda, \lambda^{-1}, a_1, \ldots, a_{n-3}$$

where λ is positive real and all a_i satisfy $|a_i| = 1$.

Nonunipotent matrices $A \in SL_2(\mathbb{Z})$ with positive eigenvalues are of Lorentz type. Any matrix $A \in GL_4(\mathbb{Z})$ with characteristic polynomial $(x-1)$ $(x^4 - 4x^3 + 4x^2 - 4x + 1)$ is of Lorentz type.

We have the following obvious result describing the classification of the $\Gamma(n+1, A)$.

PROPOSITION 1.16. *Let* $n \geqslant 1$ *be an integer and* $A, A' \in GL_n(\mathbb{Z})$. *Then:*

(i) $\Gamma(n+1, A) \cong \Gamma(n+1, A') \Longleftrightarrow A$ *is* $GL_n(\mathbb{Z})$ *conjugate to* A' *or* A'^{-1},

(ii) $\Gamma(n+1, A)$ *is commensurable with* $\Gamma(n+1, A') \Longleftrightarrow A^r$ *is* $GL_n(\mathbb{Q})$ *conjugate to* A'^s *for some* $r, s \in \mathbb{Z} \setminus \{0\}$.

THEOREM 1.17. *Let* $n \geqslant 1$ *be an integer. Then the following hold*

(i) If $\Gamma \leqslant E(n, 1)$ *is a subgroup that acts discontinuously and quasitransitively on affine space and* Γ *is not nilpotent by finite then* Γ *contains a subgroup* Γ_0 *of finite index so that* Γ_0 *is isomorphic to a group* $\Gamma(n+1, A)$ *where* $A \in GL_n(\mathbb{Z})$ *is of Lorentz type*

(ii) Every $\Gamma(n+1, A)$, *where* A *is of Lorentz type can be embedded into*

$E(n, 1)$ as a discontinuous quasitransitive group of affine transformations.

Proposition 1.12 and Theorem 1.13 give a description of the abstract commensurability classes of groups $\Gamma \leqslant E(n, 1)$ which act discontinuously and quasitransitively on affine space and that are abelian by cyclic, relative to the $GL_n(\mathbb{Q})$ conjugacy classes of certain matrices in $GL_n(\mathbb{Z})$.

The groups $\Gamma_i(n+1, k, m)$ and $\Gamma(n+1, A)$ are all torsionfree, so they are all fundamental groups of $n+1$-dimensional complete compact affine Lorentz-manifolds (space-times). Our results imply that every fundamental group of a manifold of this type is abstractly commensurable with some $\Gamma_i(n+1, k, m)$ or $\Gamma(n+1, A)$.

The Bieberbach theorems say that in each dimension there are only finitely many isomorphism types of discontinuous, quasitransitive groups of euclidean affine motions and they are all abstractly commensurable. In addition to the above classification of the abstract commensurability classes of discontinuous, quasitransitive groups of Lorentz affine motions we can add the following result on the possible isomorphism types.

THEOREM 1.18. *Let $n \geqslant 1$ be an integer, $H \leqslant E(n, 1)$ be a unipotent subgroup that acts simply transitively on affine space. Fix a subgroup $\Gamma \leqslant H$ that acts discontinuously and quasitransitively. Then the following set of groups which act discontinuously and quasitransitively*

$$\{\Delta \leqslant E(n, 1) \mid \ (i)\ \Delta \cap H = \Gamma,\ (ii)\ |\Delta : \Gamma| < \infty\}$$

falls into finitely many isomorphism classes.

Theorem 1.18 is proved in [12]. The proof used methods from [23].

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2. GROUPS ACTING DISCONTINUOUSLY AND QUASITRANSITIVELY ON AFFINE SPACE

In this section we shall give a proof of Theorem 1.3. We start off by establishing some technical results. Our proof will work by induction on the dimension of V. First of all we describe a device to divide out subspaces from V.

If the group H acts on the set S and $S' \subset S$ is a subset we write

$$\mathrm{Stab}_H(S') = \{h \in H \mid hS' \subset S'\}$$

for the stabilizer of S' in H. Suppose now that V is a finite dimensional real vectorspace and $V_0 \leqslant V$ a subspace. By r_{V_0} we denote the natural homomorphism

$$r_{V_0} : \mathrm{Stab}_{GL(V)}(V_0) \to GL(V/V_0).$$

We also have the homomorphism

$$\rho_{V_0} : \lambda^{-1}(\mathrm{Stab}_{GL(V)}(V_0)) \to \mathrm{Aff}(V/V_0)$$

$$\rho_{V_0}\begin{pmatrix} g & w^t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} V_0 + z^t \\ 1 \end{pmatrix} = \begin{pmatrix} V_0 + gz^t + w^t \\ 1 \end{pmatrix}$$

The kernel of ρ_{V_0} clearly is

$$ker\, \rho_{V_0} = \mathrm{Stab}_{\mathrm{Aff}(V)}(\mathbf{a}(V_0)) \cap \lambda^{-1}(ker\, r_{V_0}).$$

LEMMA 2.1. *Let* V *be a finite dimensional real vectorspace,* $\Gamma \leqslant \mathrm{Aff}(V)$ *a subgroup that acts discontinuously and quasitransitively on* $\mathbf{a}(V)$. *Assume that* $V_0 \leqslant V$ *is a subspace with*

(i) $\lambda(\Gamma) \leqslant \mathrm{Stab}_{GL(V)}(V_0)$,

(ii) $ker\, \rho_{V_0} \cap \Gamma$ *acts quasitransitively on* $\mathbf{a}(V_0)$.

Then $\rho_{V_0}(\Gamma)$ *acts discontinuously and quasitransitively on* $\mathbf{a}(V/V_0)$.

Proof: Clearly $\rho_{V_0}(\Gamma)$ acts quasitransitively on $\mathbf{a}(V/V_0)$.

Let K be a compact set in $\mathbf{a}(V/V_0)$. Choose a compact set $K' \leqslant \mathbf{a}(V)$ so that the image of K' in $\mathbf{a}(V/V_0)$ is K and so that the translates of $K \cap \mathbf{a}(V_0)$ under $ker\, \rho_{V_0} \cap \Gamma$ exhaust $\mathbf{a}(V_0)$. Every coset $\theta \in \rho_{V_0}(\Gamma)$ with $K \cap \theta K \neq \phi$ has then an element $\gamma \in \theta$ with $\gamma K' \cap K' \neq \phi$. Hence $\rho_{V_0}(\Gamma)$ acts discontinuously. ∎

The obvious spaces V_0 to which Lemma 2.1 can be applied are the spaces generated by the translations in Γ:

$$\mathbb{R} \cdot \left\{ w \in V \mid \begin{pmatrix} 1 & w^t \\ 0 & 1 \end{pmatrix} \in \Gamma \right\}.$$

Returning to the situation described in Theorem 1.3 we have fixed a type of $G \leqslant GL(V)$ and we are considering G-linear groups $\Gamma \leqslant \mathrm{Aff}(V)$. Let $V_0 \leqslant V$ be a subspace with

$$\lambda(\Gamma) \leqslant \mathrm{Stab}_{GL(V)}(V_0).$$

For inductive purposes it is then necessary to describe some properties of the group

$$\mathrm{Stab}_G(V_0) \leqslant G.$$

Or rather its image in $GL(V/V_0)$. For this we need a theorem of Morozov and Platonov [20, 21]. See also [6].

PROPOSITION 2.2. *Let G be a linear algebraic reductive group over $\mathbb{R}$. Let $H \leqslant G$ be a Zariski closed subgroup Assume that the unipotent radical of H is nontrivial Then the normalizer $N_G(H)$ of H in G is contained in a (proper) parabolic subgroup of G*

The following is an obvious consequence of the above result.

COROLLARY 2.3. *Let G be a linear reductive algebraic group over $\mathbb{R}$ with $rk_{\mathbb{R}} G \leqslant 1$. If H is a Zariski closed subgroup then only the following two cases are possible.*

(i) The connected component of H is reductive, with $rk_{\mathbb{R}} H \leqslant 1$.

(ii) $N_G(H)$ is a subgroup of a group P which is isomorphic to a semidirect product $P \cong S \times K$ where S is soluble and K is compact.

Proof: The parabolic subgroups of G are of the type mentioned in (ii). ■

The following proves for some groups G that a G-linear group is virtually polycyclic. The result is contained in Raghunathan [22].

LEMMA 2.4. *Let H be a connected Lie-group which is a semidirect product*

$$H = S \rtimes K$$

of a compact group K over a soluble normal group S. Then any discrete subgroup of H is virtually polycyclic.

Another result needed is the following special case of a theorem by Auslander [2].

PROPOSITION 2.5. *Let V be a finite dimensional vectorspace. $\Gamma \leqslant \mathrm{Aff}(V)$ a discrete subgroup. Then*

$$\overline{\lambda(\Gamma)}^0$$

is soluble.

Here $\overline{H}^0$ stands for the connected component of the topological closure of the subgroup $H \leqslant GL(V)$. If G is a linear semisimple Lie-group with finitely many connected components and so that $rk_{\mathbb{R}} G = 1$, we put

$$X_G = G^0/K,$$

where $K \leqslant G^0$ is a maximal compact subgroup. X_G is called the symmetric space attached to G. It is homeomorphic to $\mathbb{R}^d$ for some d. For the almost $\mathbb{R}$-simple real groups we list the dimensions of X_G from [14]:

G	$\dim X_G$
$\mathbf{O}(n, 1)$	n
$\mathbf{U}(n, 1)$	$2n$
$\mathbf{Sp}(n, 1)$	$4n$
$\mathbf{F}_4\,\mathrm{II}$	16

We prove:

PROPOSITION 2.6. *Let G be a semisimple real Lie-group with finitely many connected components and of real rank 1. Assume further that G is not isogenous to an almost direct product $K \times \mathbf{O}(2, 1)$ where K is compact. Let*

$$\rho : G \to \mathbf{GL}(V)$$

be a faithful representation (i.e. $\ker \rho = 1$). Then

$$\dim V > \dim X_G.$$

Proof. G is an almost direct product $G = K \times H$, where H is almost $\mathbb{R}$-simple with $rk_m(H) = 1$ and K is compact. ρ defines a faithful representation of H and hence a nontrivial representation of the complexification $\mathscr{H}_{\mathbb{C}}$ of the Lie algebra of H. Note that $\mathscr{H}_{\mathbb{C}}$ is simple except for H isogenous to $\mathbf{O}(3, 1)$, in this case $\mathscr{H}_{\mathbb{C}} = sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$. From the tables in [25] we see which irreducible representations of $\mathscr{H}_{\mathbb{C}}$ are real. On the other hand we get from Weyls dimension formula [15] a lower bound for the minimal dimension of an irreducible representation of $\mathscr{H}_{\mathbb{C}}$. The combination gives the following table of minimal dimensions of irreducible nontrivial modules V for the various groups H.

H isogenous to	$\dim V \geqslant$
$\mathbf{O}(n, 1)$	$n + 1$ for $n \neq 2$
$\mathbf{U}(n, 1)$	$2n + 2$
$\mathbf{Sp}(n, 1)$	$4n + 4$
$\mathbf{F}_4\,\mathrm{II}$	26

Note that $Spin(2, 1)$ has a two-dimensional representation via the exceptional isomorphism

$$Spin(2, 1) \cong SL_2(\mathbb{R}) \qquad \blacksquare$$

Proof of the Theorem 1.3. As mentioned before we proceed by induction on the dimension of V. The result is known in dimensions 1, 2, 3. See [9]. In fact in dimensions 1 and 2 our claim is more or less obvious.

Let now V be an $n \geqslant 3$ dimensional vectorspace and $\Gamma \leqslant \mathrm{Aff}(V)$ a subgroup that acts discontinuously and quasitransitively on $\mathbf{a}(V)$. If the group of translations T_Γ is nontrivial then we use Lemma 2.1, Corollary 2.3, Lemma 2.4 and the induction hypothesis to finish. We assume now that $T_\Gamma = \langle 1 \rangle$. We now distinguish two cases:

1) $\lambda(\Gamma)$ is discrete in G.

Since Γ is a finitely generated linear group, we may replace Γ by a torsionfree subgroup of finite index (Selbergs theorem [22]). Then Γ acts without fixed points on $\mathbf{a}(V)$. We may by going to a subgroup of finite index insure that Γ acts orientation preservingly. It follows that

$$\Gamma \backslash \mathbf{a}(V)$$

is a compact orientable manifold of dimension $n = \dim V$. By Poincaré-duality we see that

$$H^n(\Gamma, \mathbb{R}) \cong H^n(\pi_1(\Gamma \backslash \mathbf{a}(V)), \mathbb{R}) \cong H^n(\Gamma \backslash \mathbf{a}(V), \mathbb{R}) \cong \mathbb{R}.$$

On the other hand $\lambda(\Gamma) \leqslant G$ being discrete and torsionfree it is well known that $\lambda(\Gamma) \cong \Gamma$ acts discontinuously and without fixed points on X_G. By Proposition 2.6 Γ is also the fundamental group of the manifold

$$\Gamma \backslash X_G$$

which is of dimension $< n$. This implies that the cohomological dimension of Γ is strictly less than n. This is a contradiction. Note that since the dimension of V is $\geqslant 3$ the exceptional case in Proposition 2.6 cannot occur.

2) $\lambda(\Gamma)$ is not discrete

Then the group

$$S = \overline{\lambda(\Gamma)}^0$$

is a nontrivial connected solvable group by Auslanders theorem (2.5). We write $\widetilde{S}$ for its Zariski-closure. If $\widetilde{S}$ contains unipotent elements, that is if the unipotent radical of $\widetilde{S}$ is nontrivial then by Corollary 2.3 and Lemma 2.4 we are finished.

If $\widetilde{S}$ contains no unipotent elements it is a torus. Then the certralizer $C_G(\widetilde{S})$ of $\widetilde{S}$ in G has finite index in the normalizer $N_G(\widetilde{S})$. Let Γ_0 be a subgroup of finite index in Γ so that Γ_0 centralizes S. Put

$$\Gamma_1 = \lambda^{-1}(S) \cap \Gamma_0 \leqslant \Gamma$$

Since Γ_0 centralizes $\lambda(\Gamma_1)$ the commutator

$$[\Gamma_1, \Gamma_0]$$

is contained in the group of translations T_Γ. So Γ_1 lies in the centre of Γ_0. Take a nontrivial element

$$\gamma \in \Gamma_1.$$

Let V_γ be the eigenspace for the eigenvalue 1 of $\lambda(\gamma)$.

$$V_\gamma = \{v \in V \mid (1_n - \lambda(\gamma)) \cdot v = 0\} \leqslant V.$$

The space V_γ is left invariant by $\lambda(\Gamma_0)$ since γ is central in Γ_0. We prove now:

There is a unique coset $V_\gamma + z$ so that the affine subspace

$$(*) \qquad \begin{pmatrix} V_\gamma + z \\ 1 \end{pmatrix}$$

is left invariant by γ.

This follows easily from the fact that $\lambda(\gamma)$ is semisimple and hence $1 - \lambda(\gamma)$ is invertible on V/V_γ.

By conjugating the group Γ_0 we may assume that $z = 0$. The group Γ_0 then also leaves invariant the affine space $\mathfrak{a}(V_\gamma)$. It follows that Γ_0 acts discontinuosly and quasitransitively on $\mathfrak{a}(V_\gamma)$.

We replace now Γ_0 by a torsionfree subgroup of finite index. The manifolds

$$\Gamma_0 \backslash \mathfrak{a}(V_\gamma) \text{ and } \Gamma_0 \backslash \mathfrak{a}(V)$$

are compact and have distinct dimensions. Since Γ_0 is the fundamental group of both, the argument using the cohomological dimension of Γ_0 produces a contradiction. ■

Proof of Corollary 1.5. We may assume that $G \leqslant GL(V)$ is an algebraic group since the result is clearly true of G is not algebraic. We have already proved that Γ is virtually polycyclic. Let Δ be a torsionfree soluble subgroup of finite index in Γ. Δ is an extension of an abelian kernel by $\lambda(\Delta)$. We write $\widetilde{\lambda(\Delta)}$ for the Zariski closure of $\lambda(\Delta)$ in G. $\widetilde{\lambda(\Delta)}$ is either a torus in which case $\lambda(\Delta)$ is abelian or contains unipotent elements. If $\widetilde{\lambda(\Delta)}$ contains unipotent elements then $\lambda(\Delta)$ is contained in a parabolic P subgroup of G (Proposition 2.2). P is an almost semidirect product: $P = K \times S$ where K is compact and S is a split extension $S = U \rtimes \mathbb{R}$ where U is unipotent of class $\leqslant 2$. We now use the fact that the image of $\lambda(\Delta)$ in K/F $(F = K \cap S)$ is abelian. ■

3. UNIPOTENT SIMPLY TRANSITIVE GROUPS OF AFFINE LORENTZ-MOTIONS

In this section we shall analyse unipotent groups of affine Lorentz-motions that act simply transitively on affine space. We start off by constructing some examples.

Let $n \geqslant 1$ be an integer. We fix an $n-1$-dimensional real vector space with basis $W = \mathbb{R}^{n-1}$. On the vector space

$$\mathbb{R} \times W \times \mathbb{R}$$

we consider the quadratic form

$$q(x, w, y) = 2xy + ww^t$$

which is of signature $(n, 1)$. As in the introduction we define

$$\mathbf{O}(n, 1) = 0(q, \mathbb{R})$$

$$\mathbf{O}(n-1) = \{\sigma : W \to W \mid \sigma \text{ is linear and } \sigma\sigma^t = E_{n-1}\}.$$

$$E(n, 1) = \left\{ \begin{pmatrix} g & w^t \\ 0 & 1 \end{pmatrix} \mid g \in \mathbf{O}(n, 1),\ w \in \mathbb{R} \times W \times \mathbb{R} \right\},$$

$$Un(n, 1) = \left\{ \begin{pmatrix} 1 & -v & -\frac{1}{2} vv^t \\ 0 & E_{n-1} & v^t \\ 0 & 0 & 1 \end{pmatrix} \mid v \in W \right\} \leqslant 0(n, 1).$$

$$P(n, 1) = \left\{ \begin{pmatrix} \lambda & -v & -\frac{1}{2} \lambda^{-1} vv^t \\ 0 & \sigma & \lambda^{-1} \sigma v^t \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \mid \begin{matrix} \lambda \in \mathbb{R}^*, v \in W, \\ \sigma \in 0(n-1) \end{matrix} \right\}$$

Here E_{n-1} is the $(n-1) \times (n-1)$ identity matrix.

$Un(n, 1)$ is a maximal unipotent subgroup of $\mathbf{O}(n, 1)$. $P(n, 1)$ is a minimal parabolic subgroup of $\mathbf{O}(n, 1)$. For $u, v \in W, r, s \in \mathbb{R}$ we introduce the following elements of $E(n, 1)$ which have their linear parts in $Un(n, 1)$.

$$L(v; r, u, s) := \begin{pmatrix} 1 & -v & -\frac{1}{2} vv^t & r \\ 0 & E_{n-1} & v^t & u^t \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \in E(n, 1).$$

The following is a simple computation.

LEMMA 3.1. *Let* $W = \mathbb{R}^{n-1}$ *be an* $n-1$ *dimensional real vector space,* u, u', $v, v' \in W; r, s, r', s' \in \mathbb{R}$. *Then*

(i) $L(v; r, u, s) \cdot L(v'; r', u', s') =$

$$L\left(v + v'; r + r' - vu'^t - \frac{1}{2} s'vv^t, u + u' + s'v, s + s'\right),$$

(ii) $L(v; r, u, s)^{-1} = L\left(-v; -r - vu^t + \frac{1}{2} s\, vv^t, -u + sv, -s\right),$

(iii) $[L(v; r, u, s), \quad L(v'; r', u', s')]$

$$= L\left(0; v'u^t - vu'^t + \frac{1}{2} sv'v'^t - \frac{1}{2} s'vv^t, s'v - sv', 0\right).$$

We normalise commutators of elements g, h of a group G as:

$$[g, h] = ghg^{-1}h^{-1}.$$

Lemma 3.1 will often be used without further mention in the proofs to follow. For elements $w \in W$ and $s \in \mathbb{R}$ we also define:

$$g_w(s) := L\left(sw; -\frac{1}{6} s^3 ww^t, \frac{1}{2} s^2 w, s\right).$$

We have define $g_w(s)$ to satisfy:

$$g_w(s) = exp\ s \begin{pmatrix} 0 & -w & 0 & 0 \\ 0 & 0 & w^t & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows that the $g_w(s)$ define a unipotent 1-parameter subgroup. We are ready to introduce our groups.

DEFINITION 3.2. Let $W = \mathbb{R}^{n-1}$ be an $n-1$-dimensional vector space. We also fix a linear map

$$\psi : W \to W$$

satisfying $\psi^2 = 0$ and an element

$$w \in W$$

$$G(\psi) := \{L(\psi(u); r, u, 0) \mid u \in W, r \in \mathbb{R}\}$$

$$G(\psi, w) := \{G(\psi), G(\psi) \cdot g_w(s) \mid s \in \mathbb{R}\}.$$

The following elementary proposition describes the group theoretic structure of the $G(\psi, w)$.

PROPOSITION 3.3. *Let* $W = \mathbb{R}^{n-1}$ *be an* $n-1$*-dimensional real vector space,* $\psi : W \to W$ *a linear map with* $\psi^2 = 0$ *and* $w \in W$. *Then we have*

(i) $G(\psi)$ *and* $G(\psi, w)$ *are unipotent subgroups of* $E(n, 1)$.

(ii) $G(\psi)$ *is normal in* $G(\psi, w)$.

(iii) There is an exact diagram of groups

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathbb{R} & \xrightarrow{\theta} & G(\psi) & \to & W & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & G(\psi, w) & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathbb{R} & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

where $\theta(r) = L(0; r, 0, 0)$.

(iv) $\theta(\mathbb{R})$ *is central in* $G(\psi, w)$.

The following explain why we have introduced the $G(\psi, w)$.

PROPOSITION 3.4. *Let* $W = \mathbb{R}^{n-1}$ *be an* $n-1$*-dimensional real vector space* $\psi : W \to W$ *a linear map with* $\psi^2 = 0$ *and* $w \in W$. *Then*

(i) $G(\psi, w)$ *acts simply transitively on* $\mathbf{a}(\mathbb{R} \times W \times \mathbb{R})$.

(ii) every subgroup U *of* $E(n, 1)$ *with* $\lambda(U) \leqslant Un(n, 1)$ *that acts simply*

transitively on $\mathbf{a}(\mathbb{R} \times W \times \mathbb{R})$ *is equal to one of the* $G(\psi, w)$.

(iii) every unipotent subgroup of $E(n, 1)$ *that acts simply transitively on* $\mathbf{a}(\mathbb{R} \times W \times \mathbb{R})$ *is* $E(n, 1)$ *conjugate to a group* $G(\psi, w)$.

Proof.

(i) Follows since $G(\psi, w)$ is transitive and the stabiliser of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is trivial.

(ii) We only sketch a proof here. For a point $P \in \mathbf{a}(\mathbb{R} \times W \times \mathbb{R})$ there is a unique $\gamma \in U$ with

$$\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P.$$

Writing down the conditions for the possible γ to have a fixed point, we arrive at the required result.

(iii) Any unipotent subgroup of $\mathbf{O}(n, 1)$ is conjugate to a subgroup of $Un(n, 1)$. ∎

The classification of the $G(\psi, w)$ up to conjugacy in $E(n, 1)$ is described by:

PROPOSITION 3.5. *Let* $W = \mathbb{R}^{n-1}$ *be an* $n-1$*-dimensional real vector space* $\psi, \psi' : W \to W$ *two linear maps with* $\psi^2 = \psi'^2 = 0$ *and* $w, w' \in W$. *Then the following are equivalent*

(i) there is a $g \in E(n, 1)$ *with*

$$g\, G(\psi, w) g^{-1} = G(\psi', w')$$

(ii) there is a $\lambda \in \mathbb{R}^*$ *and a* $\sigma \in \mathbf{O}(n-1)$ *with*

$$\psi' = \lambda \sigma \psi \sigma^{-1}, \quad \lambda^2 w \sigma^t = w'.$$

Proof. Observe that if $h \in Un(n, 1)$ is nontrivial and $g \in \mathbf{O}(n, 1)$ is an element with $ghg^{-1} \in Un(n, 1)$ then it follows that $g \in P(n, 1)$.

After this apply Lemma 3.6. ∎

LEMMA 3.6. *Let* $W = \mathbb{R}^{n-1}$ *be an* $n-1$*-dimensional real vector space* $\psi : W \to W$ *a linear map and* $w \in W$. *Let furthermore*

$$g = \begin{pmatrix} \lambda & -v & -\frac{1}{2}\lambda^{-1} v v^t & r \\ 0 & \sigma & \lambda^{-1}\sigma v^t & u^t \\ 0 & 0 & \lambda^{-1} & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be an element from $P(n, 1)$. *Then:*

$$gG(\psi, w)g^{-1} = G(\lambda\sigma\psi\sigma^{-1}, \lambda^2 w\sigma^t).$$

Lemma 3.6 is proved by a simple computation.

To classify the $G(\psi, w)$ up to isomorphism we introduce the Lie algebras of these groups.

DEFINITION 3.7. Let L be a field and let $W = L^{n-1}$ be an $n-1$-dimensional L-vectorspace. For an endomorphism $\psi : W \to W$ and an element $w \in W$ we introduce the following product on the vectorspace

$$L \times W \times L$$

$$[(r, v, s), (r', v', s')] =$$

$$= (\psi(v')v^t - \psi(v)v'^t - swv'^t + s'wv^t, s\psi(v') - s'\psi(v), 0).$$

We call the so defined algebra $\mathfrak{g}_L(\psi, w)$.

The product $[\ ,\ ]$ on $\mathfrak{g}_L(\psi, w)$ is antisymmetric, it is a Lie algebra structure if and only if the map

$$W \times W \to L$$

$$(u, v) \to \psi^2(v) \cdot v'^t$$

is symmetric. This is always so in the cases which interest us here $(\psi^2 = 0)$.

If $L = \mathbb{R}$ and $\psi : W \to W$ is an endomorphism with $\psi^2 = 0$ and if $w \in V$ then $\mathfrak{g}_L(\psi, w)$ is the Lie algebra of $G(\psi, w)$. For this consider the representation

$$\theta : \mathfrak{g}_{\mathbb{R}}(\psi, w) \to End(\mathbb{R} \times W \times \mathbb{R} \times \mathbb{R})$$

$$\theta : (r, u, s) \to \begin{pmatrix} 0 & -\psi(u) - sw & 0 & r \\ 0 & 0 & \psi(u)^t + sw^t & u^t \\ 0 & 0 & 0 & s \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have

$$exp(\theta(\mathfrak{g}_{\mathbb{R}}(\psi, w)) = G(\psi, w).$$

Over any field L the algebras $\mathfrak{g}_L(\psi, w)$ can be classified up to L-isomorphism. We give only a partial answer here.

DEFINITION 3.8. Let $n \geqslant 1$, k be integers with $0 \leqslant k \leqslant (n-1)/2$. For a field L let W be a vectorspace of dimension $n-1$ with basis

$$e_1, f_1, \ldots, e_k, f_k, e_{2k+1}, \ldots, e_{n-1}.$$

Let ψ_k be the endomorphism of W defined by

$$\psi_k(e_i) = 0, \ \psi_k \ (f_i) = e_i, \quad i = 1, \ldots, k.$$

For any $0 \leqslant k \leqslant (n-1)/2$ we define the Lie-algebra,

$$\mathscr{L}_L^2(n+1, k) = \mathfrak{g}_L(\psi_k, 0)$$

for $k \neq (n-1)/2$ put:

$$\mathscr{L}_L^2(n+1, k) = \mathfrak{g}_L(\psi_k, e_{2k+1}).$$

It is easy to prove the following classification result. See also Proposition 6.2.

PROPOSITION 3.9. *Let K be a field and $W = K^{n-1}$ an $n-1$-dimensional K-vectorspace. Let $\psi : W \to W$ be an endomorphism and $w \in W$. Then the Lie algebra $\mathfrak{g}_L(\psi, w)$ is isomorphic to exactly one of the above defined $\mathscr{L}_L^i(n+1, k)$.*

In table 1 we have computed a multiplication table for the Lie algebras $\mathscr{L}_{\mathbb{R}}^i(n+1, k)$. Proposition 3.9 and Proposition 3.4 then imply Theorem 1.8.

Table 1

	$\mathscr{L}^1(n+1,k)$	$\mathscr{L}^2(n+1,k)$	for
$[\tau,\zeta]=$	0	0	
$[\tau,e_i]=$	0	$-\delta_{2k+1,i}\cdot\zeta$	all i
$[\tau,f_j]=$	$-e_j$	$-e_j$	$1\leqslant j\leqslant k$
$[e_i,e_j]=$	0	0	all i,j
$[e_i,f_j]=$	$\delta_{ij}\cdot\zeta$	$\delta_{ij}\cdot\zeta$	$1\leqslant j\leqslant k$
$[f_i,f_j]=$	0	0	$1\leqslant i,j\leqslant k$
$[\zeta,e_i]=$	0	0	all i
$[\zeta,f_j]=$	0	0	$1\leqslant j\leqslant k$

$\delta_{ij}=\begin{matrix}1 & \text{for } i=j\\ 0 & \text{for } i\neq j\end{matrix}$

4. SIMPLY TRANSITIVE GROUPS OF EUCLIDEAN MOTIONS

We shall describe here the simply transitive groups of euclidean motions in a way which will be convenient for us in the next chapter.

Let $V=\mathbb{R}^n$ be a real vector space with basis. On the chosen basis we consider the quadratic form

$$q=x_1^2+\ldots+x_n^2.$$

We write $\mathbf{O}(n)$ for its orthogonal group. We let furthermore

$$\mathscr{E}(n)=\left\{\begin{pmatrix}g & v^t\\ 0 & 1\end{pmatrix}\,\middle|\, g\in\mathbf{O}(n), v\in V\right\}$$

be the group of euclidean motions on $\mathbf{a}(V)$.

We shall now define some subgroups of $\mathscr{E}(n)$ which act simply transitively on $\mathbf{a}(V)$.

DEFINITION 4.1. Let $0\leqslant d<n\geqslant 1$ be integers. We put

$$V_d=\{(x_1,\ldots,x_n)\in V\mid x_1=\ldots=x_d=0\},$$
$$V^d=\{(x_1,\ldots,x_n)\in V\mid x_{d+1}=\ldots=x_n=0\}.$$
$$\epsilon: V_d\to\mathbf{O}(d)$$

be a continuous homomorphism with discrete kernel. We define the group

$$\mathscr{E}(\epsilon, d) = \left\{ \begin{pmatrix} \epsilon(u) & 0 & w^t \\ 0 & E & u^t \\ 0 & 0 & 1 \end{pmatrix} \middle| \, u \in V_d, \, w \in V^d \right\}.$$

The set $\mathscr{E}(\epsilon, d)$ is a subgroup of $\mathscr{E}(n)$. Note that for an ϵ to exist it is necessary that $[(n-d)/2] \geqslant d$. Here $[x]$ is the biggest integer smaller than the real number x.

PROPOSITION 4.3. *Let* $0 \leqslant d, d' < n > 0$ *be integers.*

(i) The groups $\mathscr{E}(\epsilon, d)$ *for homomorphisms*

$$\epsilon \; : V_d \to \mathbf{O}(d)$$

act simply transitively on $\mathbf{a}(V)$.

(ii) If $\mathscr{E}(\epsilon, d), \mathscr{E}(\epsilon', d')$ *are two groups attached to homomorphisms with discrete kernels*

$$\epsilon : V_d \to \mathbf{O}(d), \; \epsilon' : V_{d'} \to \mathbf{O}(d')$$

then $\mathscr{E}(\epsilon, d)$ *is* $\mathscr{E}(n)$ *conjugate to* $\mathscr{E}(\epsilon', d')$ *if and only if:*

$$d = d' \quad \text{and} \qquad g \in \mathbf{O}(d), \, h \in \mathbf{O}(n-d)$$

$$\text{with } \; \epsilon'(uh^{-1}) = g\epsilon(u)\, g^{-1}.$$

The proof of Proposition 4.2 is easy. The following result shows that the $\mathscr{E}(\epsilon, d)$ are essentially all simply transitive subgroups of $\mathscr{E}(n)$.

PROPOSITION 4.3: *Let* $n \geqslant 0$ *be an integer. Suppose that* $H \leqslant \mathscr{E}(n)$ *is a group that acts simply transitively on affine space. Then there is an* $0 \leqslant d < n$ *and a homomorphism with discrete kernel* $\epsilon : V_d \to \mathbf{O}(d)$ *so that* H *is conjugate in* $\mathscr{E}(n)$ *to* $\mathscr{E}(\epsilon, d)$.

To prove Proposition 4.3 we need:

LEMMA 4.4. *Let* $n \geqslant 0$ *be an integer. Suppose that* $H \leqslant \mathscr{E}(n)$ *is a commutative group that acts simply transitively on affine space. Then* $H = T_{\epsilon(n)}$, *that is* H *is the full group of translations.*

Proof. We proceed by induction on n. The result is claar for $n = 1$. Suppose now that $n \geqslant 2$. The group H acts by conjugation on $T_{\epsilon(n)}$ and hence on $V = V_{\epsilon(n)}$. It leaves invariant the spaces V_H^0 and $V_H^{0\perp}$. Note that since H is commutative, H acts trivially on V_H^0.

The group $\lambda(H)$ is a commutative connected subgroup of $\mathbf{O}(n)$, hence we have

$$\dim(\lambda(H)) \leqslant \left[\frac{n}{2}\right].$$

It follows that

$$\dim(V_H^{0\perp}) < n.$$

The stabilizer

$$H_1 = \mathrm{stab}_H(\mathbf{a}(V_H^{0\perp}))$$

acts simply transitively on $\mathbf{a}(V_H^{0\perp})$. The result follows by induction. ■

Proof of Proposition 4.2. The group H acts by conjugation on the space of all translations $T_{\epsilon(n)}$ and hence on $V = V_{\epsilon(n)}$. It leaves the spaces V_H^0 and $V_H^{0\perp}$ invariant. We put

$$H_1 = \mathrm{stab}_H(\mathbf{a}(V_H^{0\perp})).$$

H_1 acts simply transitively on $\mathbf{a}(V_H^{0\perp})$. By a theorem of Auslander [1] H_1 is a connected soluble group. Hence $\lambda(H_1) \leqslant \mathbf{O}(n)$ is commutative. The kernel

$$H_2 = ker(\lambda : H_1 \to \mathbf{O}(n))$$

is subgroup of translations in $V_H^{0\perp}$, hence H_2 is discrete. H_2 acts by conjugation on the discrete group H_2. Since $\lambda(H_1)$ is compact H_1 has to centralize H_2. For a given $g \in H_1$ we consider the map

$$[g,\ \ ] : H_1 \to H_2$$
$$[g,\ \ ] : h \to [g, h].$$

The commutator subgroup of H_1 being central, the map $[g,\ \ ]$ is a homomorphism of the connected group H, into the discrete group H_2. It follows that H_1 is commutative. By Lemma 4.3 H_1 acts by translations on $\mathbf{a}(V_H^{0\perp})$. We may consider the resulting homomorphism

$$\epsilon : V_H^{0\perp} \to H_1 \to \mathbf{O}(n),$$

where an element v is mapped to the corresponding translation in H_1.

The image of ϵ stabilizes $V_H^{0\perp}$ and leaves $V_H^{0\perp}$ pointwise fixed. Finally we conjugate the pair of subspaces V_H^0 and $V_H^{0\perp}$ by an orthogonal matrix into the standard pair. ■

The above shows that the groups $H \leqslant \mathscr{E}(n)$ that act simply transitively on affine space are all split extensions

$$0 \to \mathbb{R}^a \to H \to \mathbb{R}^b \to 0$$

with $a + b = n$ and $\mathbb{R}^b$ acts orthogonally on $\mathbb{R}^a$. Conversely, it is clear that every split extension of this type can be embedeed into $\mathscr{E}(n)$ as a simply transitive group of euclidean motions.

5. NONUNIPOTENT SIMPLY TRANSITIVE GROUPS OF AFFINE LORENTZ-MOTIONS

Here we shall describe the simply transitive groups of affine Lorentz-motions which contain nonunipotent elements. To do this we introduce the following subgroups of $P(n, 1)$. Here $W = \mathbb{R}^{n-1}$ is an $(n-1)$-dimensional real vector space with chosen basis.

$$D(n, 1) = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda \in \mathbb{R}^*, \ \sigma \in \mathrm{O}(n-1) \right\},$$

$$\hat{p}(n, 1) = \left\{ \begin{pmatrix} 1 & -v & -\frac{1}{2} vv^t \\ 0 & \sigma & \sigma v^t \\ 0 & 0 & 1 \end{pmatrix} \middle| \sigma \in \mathrm{O}(n-1), \ v \in W \right\}$$

PROPOSITION 5.1. *Let $n \geqslant 1$ be an integer.*

(i) Let $H \leqslant E(n, 1)$ be a subgroup that acts simply transitively on affine space. Then H is $E(n, 1)$ conjugate to a group H_1 with $\lambda(H_1) \leqslant P(n, 1)$.

(ii) Let $H \leqslant E(n, 1)$ be a subgroup that acts simply transitively on affine space. If H satisfies $\lambda(H) \leqslant P(n, 1)$, then either $\lambda(H) \leqslant \hat{p}(n, 1)$ or $\lambda(H)$ is $P(n, 1)$ conjugate to a subgroup of $D(n, 1)$.

Proof.

(i) Since H is a soluble group the Zariski closure $\lambda(H)$ of $\lambda(H)$ is also soluble. It is then conjugate to a subgroup of the minimal parabolic $P(n, 1)$ of $\mathrm{O}(n, 1)$.

(ii) Let U be the unipotent radical of the Zariski closure of H. By [1] U also acts simply transitively on affine space. So by proposition 3.4 it is equal to a $G(\psi, w)$. H normalizes U and hence by lemma 3.6 it follows that

$H \leqslant \hat{p}(n, 1)$, unless $\psi \equiv 0$ and $w = 0$. This being the case $\lambda(H)$ is a torus and as such $P(n, 1)$-conjugate to a subgroup of $D(n, 1)$. ■

We shall now separately discuss the two cases $(\lambda(H) \leqslant \hat{p}(n, 1)$ or $D(n, 1))$ which have come up in Proposition 5.1.

A) Groups with $\lambda(H) \leqslant D(n, 1)$

We keep here the conventions of section 4 concerning the coordinate subspaces W_d and W^d of the vectorspace $W = \mathbb{R}^{n-1}$.

DEFINITION 5.2. Let $0 \leqslant d < n - 1 \geqslant 0$ be integers, and $W = \mathbb{R}^{n-1}$ a real vectorspace. Let

$$\eta : \mathbb{R} \times W_d \times \mathbb{R} \to \mathbf{O}(d)$$

be a homomorphism so that η restricted to $(0, W_d, 0)$ has discrete kernel. Define

$$D_1(d, \eta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & r \\ 0 & \eta(r, w, s) & 0 & 0 & v^t \\ 0 & 0 & E & 0 & w^t \\ 0 & 0 & 0 & 1 & s \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid r, s \in \mathbb{R}, w \in W_d, \ v \in W^d \right\}$$

Let $0 \leqslant d_2 \leqslant d_1 < n - 1 \geqslant 1$ be integers, and $W = \mathbb{R}^{n-1}$ a real vectorspace. Let

$$\mu : W_{d_1} \to \mathbf{O}(d_1),$$

be a homomorphism with discrete kernel so that the eigenspace for the trivial character of the torus

$$\left\{ \begin{pmatrix} \mu(w) & 0 \\ 0 & E \end{pmatrix} \mid w \in W_{d_1} \right\} \leqslant \mathbf{O}(n-1)$$

is W_{d_2}. Let furthermore

$$\tau : W_{d_2} \to \mathbb{R}$$

be a homomorphism. Define

$$D_2(d_1, \mu, \tau) = \left\{ \begin{pmatrix} e^{\tau((w_2, w_1))} & 0 & 0 & 0 & 0 & r \\ 0 & & & 0 & 0 & v^t \\ & \multicolumn{2}{c}{\mu(w_1)} & & & \\ 0 & & & 0 & 0 & w_2^t \\ 0 & 0 & 0 & 1 & 0 & w_1^t \\ 0 & 0 & 0 & 0 & e^{-\tau((w_2, w_1))} & s \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} r, s \in \mathbb{R}, \\ w_1 \in W_{d_1}, \\ (w_2, w_1) \in W_{d_2}, \\ v \in W^{d_3} . \end{array} \right\}$$

The following is the reason for the above definitions.

PROPOSITION 5.3. *Let* $n \geqslant 1$ *be an integer.*

(i) The sets $D_1(d, \eta)$ *and* $D_2(d_1, \mu, \tau)$ *attached to data* d, η *and* d_1, μ, τ *as in definition 5.2 are subgroups of* $E(n, 1)$ *that act simply transitively on* $n+1$*-dimensional affine space.*

(ii) Let $n \geqslant 1$ *be an integer. Let* $H \leqslant E(n, 1)$ *be a subgroup that acts simply transitively on affine space and that satisfies* $\lambda(H) \leqslant D(n, 1)$. *Then* H *is* $E(n, 1)$*-conjugate to one of the groups* $D_1(d, \eta)$ *or* $D_2(d_1, \mu, \tau)$ *with appropriate data.*

Proof. The group H may be parametrized as

$$H = \left\{ \begin{pmatrix} e^{\tau(r, w, s)} & 0 & 0 & r \\ 0 & \sigma(r, w, s) & 0 & w^t \\ 0 & 0 & e^{-\tau(r, w, s)} & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; r, s \in \mathbb{R}, w \in W \right\},$$

with functions

$$\tau : \mathbb{R} \times W \times \mathbb{R} \to \mathbb{R}$$

$$\sigma : \mathbb{R} \times W \times \mathbb{R} \to O(n-1).$$

τ, σ have to satisfy certain functional equations coming from the fact that H is a group. These equations together with the fixed point freeness of H imply $\tau(r, w, s) = \tau(0, w\sigma(r, 0, s), 0)$. This implies that H is the product of the following two of its subgroups:

$$H_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & r \\ 0 & \sigma(r, 0, s) & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; r, s \in \mathbb{R} \right\}$$

$$H_2 = \left\{ \begin{pmatrix} e^{\tau(0,w,0)} & 0 & 0 & 0 \\ 0 & \sigma(0,w,0) & 0 & w^t \\ 0 & 0 & e^{-\tau(0,w,0)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, w \in W \right\}.$$

H_2 acts as a simply transitive group of euclidean motions on $\mathfrak{a}(0 \times W \times 0) \leqslant \leqslant \mathfrak{a}(\mathbb{R} \times W \times \mathbb{R})$. We use section 4 to find d and the homomoprhism ϵ with discrete kernel.

Assume then $\tau(0, w, 0) = 0$ for all $w \in W$. The computation of the commutators $[H_1, H_2]$ shows that H has to be conjugate to a group of type $D_1(d, \eta)$.

If $\tau(0, w, 0) \neq 0$ for some $w \in W$, the computation of the commutators $[H_1, H_2]$ shows that $\sigma(r, 0, s) = 1$ for all $r, s \in \mathbb{R}$. After this the same analysis of commutators shows that H is conjugate to a group of type $D_2(d_1, \mu, \tau)$. Note that there is an element $w \in W$ so that $\sigma(0, w, 0) - E_{n-1}$ is invertible on the orthogonal complement of the eigenspace of the trivial character of the torus

$$\{\sigma(0, w, 0) \mid w \in W\}. \qquad \blacksquare$$

Note that there is a certain obvious overlap amongs the groups of type $D_1(d, \eta)$ and $D_2(d_1, \mu, \tau)$. We have not included the classification of the $E(n, 1)$ conjugacy classes of the above groups. This can be done by elementary means but is rather messy and includes the introduction of finer invariants (such as the eigenspaces) of the homomorphisms ϵ and η, τ.

Note that the Lie groups H of type $D_1(d, \eta)$ are all split extensions

$$0 \to \mathbb{R}^a \to H \to \mathbb{R}^b \to 0$$

where $b > 2$ and $\mathbb{R}^b$ acts on $\mathbb{R}^a$ through a homomorphism $\epsilon : \mathbb{R}^b \to \mathrm{O}(a)$ such that the dimension of the connected component of the kernel of ϵ does not exceed 2. The groups H of type $D_2(d_1, \mu, \tau)$ are split extension

$$0 \to \mathbb{R}^a \to H \to \mathbb{R}^b \to 0$$

where $a > 2$ and $\mathbb{R}^b$ acts on $\mathbb{R}^a$ through a homomorphism $\epsilon : \mathbb{R}^b \to \mathbb{R}^* \times \mathrm{O}(a-2) \to GL_a(\mathbb{R})$ where $\mathbb{R}^*$ acts trivially apart from 1-dimensional eigenspaces for the identical character and its inverse.

All these groups can easily be classified up to isomorphism.

B) Groups with $\lambda(H) \leqslant \hat{P}(n, 1)$

We introduce the following subspaces of our real vectorspace $\mathbb{R} \times W \times \mathbb{R}$:

$$u = \{(r, 0, 0) \mid r \in \mathbb{R}\}, \quad \tilde{u} = \{(r, w, 0) \mid r \in \mathbb{R}, w \in W\},$$

together with the groups:

$$R = \left\{\begin{pmatrix} E & v^t \\ 0 & 1 \end{pmatrix} \mid v \in u\right\}, \ \tilde{R} = \left\{\begin{pmatrix} g & v^t \\ 0 & 1 \end{pmatrix} \mid g \in \hat{P}(n, 1), v \in \tilde{u}\right\}.$$

We start off with a group $H < E(n, 1)$ that acts simply transitively on affine space and satisfies $\lambda(H) \leqslant \hat{P}(n, 1)$. Note that the groups $H_0 = H \cap \tilde{R}$ is normal in H and satisfies $H/H_0 \cong \mathbb{R}$.

We write H for the Zariski-closure of H and U_H for its unipotent radical. By a theorem of Auslander U_H also acts simply transitively on affine space. Since $\lambda(U_H) \leqslant Un(n, 1)$ by section 3 we find $U_H = G(\psi_H, w_H)$ for appropriate ψ_H and w_H. The following gives a normal form for the subgroups $H_0 \leqslant H$ which are of codimension 1:

PROPOSITION 5.4. *Let H be a group with $H \leqslant E(n, 1)$ that acts simply transitively on affine space and satisfies $\lambda(H) \leqslant \hat{P}(n, 1)$. H can be conjugated by an element of $E(n, 1)$ so that $H_0 = H \cap \tilde{R}$ is of one of the following shapes*

(a)

$$\left\{\begin{pmatrix} 1 & 0 & -\psi(w_2, w_3) & 0 & -\frac{1}{2}\psi(w_2, w_3)\cdot\psi(w_2, w_3)^t & r \\ 0 & \epsilon(w_3) & 0 & 0 & 0 & w_1^t \\ 0 & 0 & E & 0 & \psi(w_2, w_3)^t & w_2^t \\ 0 & 0 & 0 & E & 0 & w_3^t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},\right.$$

$w_1 \in W^{d_1}$, $(w_2, w_3) \in W_{d_1}$, $w \in W_{d_2}$, $r \in \mathbb{R}.\}$

Here $\epsilon : W_{d_2} \to O(d_1)$ is a homomorphism with discrete kernel and $\psi : W_{d_1} \to W_{d_1} \cap W^{d_1}$ is a homomorphism with $\psi^2 = 0$, for appropriate integers d_1, d_2 with $d_1 \leqslant d_2$.

(b)

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & r \\ 0 & \epsilon(w_2, r) & 0 & 0 & w_1^t \\ 0 & 0 & E & 0 & w_2^t \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid w_1 \in W_{d_1}, w_2 \in W^{d_1}, \ r \in \mathbb{R} \right\}$$

Here $\epsilon : \mathbb{R} \times W^{d_1} \to \mathbf{O}(d_1)$ *is a homomorphism and* d_1 *is an appropriate integer.*

Proof. We define $H_1 = (H \cap G(\psi_H))^0$. H_1 is a connected group hence there is a subvectorspace $V \leqslant W$ so that

$$H_1 R = \left\{ \begin{pmatrix} 1 & -\psi(v) & -\frac{1}{2}\psi(v)\psi(v)^t & r \\ 0 & E & \psi(v)^t & v^t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid r \in \mathbb{R}, v \in V \right\}$$

The group

$$H_2 = \mathrm{stab}_H(\mathbf{a}(V^\perp \cap \tilde{u}))$$

acts simply transitively on $\mathbf{a}(V^\perp \cap \tilde{u})$ and satisfies

$$[H_2, H_2] \leqslant R.$$

An analysis analogous to that in section 3 together with some commutator computations finishes the result. ■

Note that groups of type (a), (b) in the above proposition all act simply transitively on the affine subspace $\mathbf{a}(W)$.

Proof of Theorem 1.9. This result can now be read of from the normal forms in Proposition 5.3 and 5.4. Under (ii) we have given an internal description of the groups $D_1(d, \eta)$. (iii) corresponds to the groups $D_2(d, \mu, \tau)$. (iv), (v) corresponds to the cases (a), (b) in Proposition 5.4 respectively. Here a commutator computation shows how the quotient $H/H_0 = \mathbb{R}$ acts on H_0.

6. GROUPS ACTING DISCONTINUOUSLY AND QUASITRANSITIVELY ON LORENTZ SPACE

We shall prove here our results on the abstract commensurability classes of groups that act discontinuously and quasitransitively on Lorentz space. The theorem of Fried and Goldman (1.5) reduces this problem to the study of the abstract commensurability classes of lattices in groups of affine Lorentz transformations that act simply transitively on affine space.

We first treat the case of unipotent groups. Let $U \leqslant E(n, 1)$ be a unipotent groups acting simply transitively on affine space and let $\mathscr{U}$ be its Lie algebra. Let $\Gamma \leqslant U$ be a subgroup that acts discontinuously and quasitransitively on affine space. We write

$$M_{\mathbb{Q}}(\Gamma) \leqslant U$$

for the Malcev completion of Γ. $M_{\mathbb{Q}}(\Gamma)$ can be described as the radicable hull of Γ, [16], [24]. We put

$$\mathscr{M}_{\mathbb{Q}}(\Gamma) := exp^{-1}(M_{\mathbb{Q}}(\Gamma)).$$

$\mathscr{M}_{\mathbb{Q}}(\Gamma)$ is a rational Lie subalgebra of $\mathscr{U}$ with the property that the map

$$\mathbb{R} \otimes_{\mathbb{Q}} \mathscr{M}_{\mathbb{Q}}(\Gamma) \to \mathscr{U}$$

is an isomorphism. That is $\mathscr{U}_{\mathbb{Q}}(\Gamma)$ is a $\mathbb{Q}$-form of the real Lie algebra $\mathscr{U}$.

Given a Lie algebra $\mathscr{M}$ over $\mathbb{Q}$ and an isomorphism of Lie algebras

$$\mathbb{R} \otimes_{\mathbb{Q}} \mathscr{M} \xrightarrow{\theta} \mathscr{U}$$

we may construct a group $\Gamma \leqslant U$ in the following way. Choose, as is always possible, a $\mathbb{Z}$-lattice $\mathscr{M}_{\mathbb{Z}} \leqslant \mathscr{M}$ invariant under the bracket and define

$$\Gamma = \langle exp(\theta(\mathscr{M}_{\mathbb{Z}}))\rangle$$

to be the group generated by the set $exp(\theta(\mathscr{M}_{\mathbb{Z}}))$. Γ acts discontinuously and quasitransitively on affine space and has the property $\mathscr{M}_{\mathbb{Q}}(\Gamma) = \theta(\mathscr{M})$. The set $exp(\theta(\mathscr{M}_{\mathbb{Z}}))$ is enclosed as

$$\Gamma_1 \leqslant exp(\theta(\mathscr{M}_{\mathbb{Z}})) \leqslant \Gamma_2$$

between the groups Γ_1, Γ_2 that act discontinuously and quasitransitively on affine space and satisfy $|\Gamma_2/\Gamma_1| < \infty$, [19].

It is well known that two torsionfree finitely generated nilpotent groups are abstractly commensurable if and only if the Lie algebras $\mathscr{M}_{\mathbb{Q}}(\Gamma_1)$, $\mathscr{M}_{\mathbb{Q}}(\Gamma_1)$ of their rational Malcev completions are isomorphic, [13]. So, to describe the abstract commensurability classes of the groups Γ we have to find the $\mathbb{Q}$-forms of the real Lie algebras $\mathscr{L}^1(n+1, k)$ and $\mathscr{L}^2(n+1, k)$. To do this we introduce a class of Lie algebras which might be of independent interest.

DEFINITION 6.1. Let L be a field and let $k, m \geqslant 0$ be integers. Let

$$E = L^k, \quad F = L^k, \quad G = L^m$$

be L-vectorspaces of the indicated dimensions. We write e_i, f_i, g_i for the canonical basis elements of E, F, G respectively and define the linear isomorphism

$$: F \to E; \qquad : f_i = e_i, \quad i = 1, \ldots, k.$$

Put

$$W = E \oplus F \oplus G.$$

Let

$$\varphi = \varphi_E + \varphi_F + \varphi_G : W = E \oplus F \oplus G \to L$$

be a linear map.

Let furthermore $S \in Sym_k(L)$ be a symmetric $k \times k$ matrix with entries in L and $\det(\ ) \neq 0$. On the L vectorspace of dimension $2k + m + 2$

$$L \oplus W \oplus \dot{L}$$

we define the product

$$[(r, u, s), (r', u', s')] =$$
$$= (\hat{f}'S\, e^t - \hat{f}\, S\, e'^t + s'\varphi(u) - s\varphi(u'),\ s'\hat{f} - s\hat{f}', 0)$$

where

$$r, s, r', s' \in L,\ u = e + f + g,\ u' = e' + f' + g' \in W.$$

We write $\mathfrak{g}_L(k, m, \varphi, S)$ for the vectorspace $L \oplus W \oplus L$ with the product $[\ ,\ ]$.

$\mathfrak{g}_L(k, m, \varphi, S)$ is always a nilpotent Lie algebra of nilpotency class $\leqslant 3$. Writing $z = (1, 0, 0)$ and $\tau = (0, 0, 1)$, we find the following defining relations for $\mathfrak{g}_L\ (k, m, \varphi, (s_{ij}))$.

$$(*) \qquad \begin{array}{lll}
[\tau, e_i] = -\varphi_E(e_i) \cdot z & & i = 1, \ldots, k, \\
[\tau, f_i] = -e_i - \varphi_F(f_i) \cdot z & & i = 1, \ldots, k, \\
[\tau, g_i] = -\varphi_G(g_i) \cdot z & & i = 1, \ldots, m, \\
[e_i, e_j] = 0, & [f_i, f_j] = 0 & i, j = 1, \ldots, k, \\
[e_i, g_j] = 0, & [f_i, g_j] = 0 & i = 1, \ldots, k; j = 1, \ldots, m,
\end{array}$$

$$[g_i, g_j] = 0, \qquad i, j = 1, \ldots, m,$$

$$[e_i, f_j] = s_{ij} \cdot z \qquad i, j = 1, \ldots, k.$$

Note that

$$\mathfrak{g}_L(\psi, w) \cong \mathfrak{g}_L(k, m, \varphi, E_k)$$

where $\mathfrak{g}_L(\psi, w)$ is one of the Lie algebras defined in section 3 and k, m, φ are appropriately chosen. Similarly

$$\mathscr{L}_{\mathbb{Q}}^{1,2}(n+1, k, S) = \mathfrak{g}_{\mathbb{Q}}(k, n-1-2k, \varphi_{1,2}, S)$$

for appropriate $\varphi_{1,2}$.

PROPOSITION 6.2. *Let L be a field and let $\mathfrak{g}_L(k, m, \varphi, S)$ and $\mathfrak{g}_L(k', m', \varphi', S')$ be two of the Lie algebras defined in 6.1. Then the following are equivalent:*

(i) $\mathfrak{g}_L(k, m, \varphi, S)$ is isomorphic to $\mathfrak{g}_L(k', m', \varphi', S')$.

(ii) $k = k'$, $m = m'$, and there are $X \in GL_k(L)$ and $Y \in GL_m(L)$, and $\alpha \in L \setminus \{0\}$ so that:

$$S' = \alpha X S X^t, \quad \varphi'_G = \varphi_G \cdot Y.$$

Proof. The proof is obtained by writing down a linear isomprphism θ on the natural basis given in definition 6.1. θ is a Lie algebra isomorphism if certain relations between the entries of θ hold. An elementary analysis of these relations implies proposition 6.2. ■

Proposition 6.2 can be used to classify the Lie algebras $\mathfrak{g}_L(k, m, \varphi, S)$. For the linear map φ we have only to consider two possibilities.
Next, we have to classify symmetric nondegenerate $k \times k$ matrices S up to the equivalence relation:

$$S' \sim S \Longleftrightarrow S' = \alpha \cdot XSX^t \text{ with } \alpha \in L^*, \ X \in GL_k(L).$$

Over the field L of real number S and S' are equivalent if they have the same or opposite signatures. Two symmetric matrices $S, S' \in Sym_k(\mathbb{Q})$ can only be equivalent over $L = \mathbb{Q}$ if they have the same or opposite signatures. We note here the following obvious consequence of the theorem of Hasse and Minkowski.

PROPOSITION 6.3. *Let $k \geqslant 1$ be an integer and let $S, S' \in Sym_k(\mathbb{Q})$ be positive definite symmetric matrices. The following are equivalent*

(i) $S' \sim S$ over $\mathbb{Q}$,

(ii) $\exists\, \alpha \in \mathbb{Q}$ with $\alpha > 0$ such that

$$(\alpha^k \cdot \det S)(\det S')^{-1} \in \mathbb{Q}^{*2} \text{ and}$$

$$C_p(S) \cdot \left(\frac{\det(S), \alpha}{p}\right)\left(\frac{\alpha, \alpha}{p}\right)^{\frac{k(k-1)}{2}} = C_p(S') \text{ for all primes } p.$$

Here $C_p(S)$ *and* $\left(\frac{a, b}{p}\right)$ *are the usual Hasse symbols at p.*

PROPOSITION 6.4. *Let* $K \leqslant L$ *be fields. Let* $\mathfrak{g}_L(k, m, \varphi, S)$ *be a Lie algebra as defined in 6.1. Let* $\mathscr{H} \leqslant \mathfrak{g}_L(k, m, \varphi, S)$ *be a K-Lie algebra so that the natural map*

$$L \otimes_K \mathscr{H} \to \mathfrak{g}_L(k, m, \varphi, S)$$

is an isomorphism. Then there is a K-linear map φ' *and an* $S' \in Sym_k(K)$ *so that*

$$\mathscr{H} \cong \mathfrak{g}_K(k, m, \varphi', S')$$

as K - Lie algebras.

Proof. The result is clear for $k = 0$. So we assume that $k \geqslant 1$. We choose in $\mathfrak{g}_L(k, m, \varphi, S)$ its natural basis z, e_i, f_i, g_i, τ which satisfy the relations (*). We shall construct now in $\mathscr{H}$ a K-basis which also satisfies our relations (*).

We have

$$[\mathscr{H}, [\mathscr{H}, \mathscr{H}]] = K \cdot \hat{z}$$

with $\hat{z} = \pi z$ for some $\pi \in L$. This follows since the Lie algebra $\mathfrak{g}_L(k, m, \varphi, S)$ has a similar property. Furthermore the commutator algebra

$$[\mathscr{H}, \mathscr{H}]$$

has dimension $k + 1$ and is contained in $L \cdot \langle z, e_1, \ldots, e_k \rangle$. We choose a basis $\hat{z}, \hat{e}_1, \ldots, \hat{e}_k$ of $[\mathscr{H}, \mathscr{H}]$. The elements $\hat{e}_i$ satisfy $[\hat{e}_i, \hat{e}_j] = 0$ for $i, j = 1, \ldots, k$ since

$$[[\mathscr{H}, \mathscr{H}], [\mathscr{H}, \mathscr{H}]] = 0.$$

The center of the Lie algebra $\mathscr{H}/[[\mathscr{H}, \mathscr{H}], \mathscr{H}]$ has dimension $k + m$ and its preimage in $\mathscr{H}$ is contained in $L \cdot \langle z, e_1, \ldots, e_k, g_1, \ldots, g_k \rangle$. We add elements to obtain a basis:

$$z, \hat{e}_1, \ldots, \hat{e}_k, \hat{g}_1, \ldots, \hat{g}_k$$

of this space. Clearly the e_i, g_j all commute with each other. We choose

$$\hat{\tau} = \hat{v}_0 + \pi_0 \tau$$

$$\hat{f}_1 = \hat{v}_1 + \pi_1 \tau$$

$$\vdots$$

$$\hat{f}_k = \hat{v}_k + \pi_k \tau$$

with $\pi_i \in L$, $\pi_0 \neq 0$, $\hat{v}_i \in L \langle z, e_1, \ldots, e_k, g_1, \ldots, g_m \rangle$ so that $\hat{z}, \hat{e}, \ldots, \hat{e}_k, \hat{f}_1, \ldots, \hat{f}_k, \hat{g}_i, \ldots, \hat{g}_m, \hat{\tau}$ is a basis of $\mathscr{H}$.

By a simple computation we find

$$[\hat{f}_i, \hat{f}_j] = \pi_i \pi_0^{-1}([\hat{\tau}, \hat{f}_i]) - \pi_j \pi_0^{-1}([\hat{\tau}, \hat{f}_j]).$$

This shows that the $[\hat{\tau}, \hat{f}_1], \ldots, [\hat{\tau}, \hat{f}_k]$ are L-linearly independent and that $\pi_i \pi_0^{-1} \in K$ for $i = 1, \ldots, k$.

We change our basis so that $\pi_1 = \ldots = \pi_k = 0$ and so that

$$(**) \qquad [\hat{\tau}, \hat{f}_i] = -\hat{e}_i + \lambda_i \hat{z}; \quad i = 1, \ldots, k$$

with $\lambda_i \in K$.

We now show how to enforce $[\hat{f}_i, \hat{f}_j] = 0$ for $i, j = 1, \ldots, k$. Consider the following bilinear map which is induced by the commutator

$$B : [\mathscr{H}, \mathscr{H}] \,/\, k \cdot \hat{z} \times F/E \to k\hat{z}$$

Here $F = K \langle \hat{z}, \hat{e}_1, \ldots, \hat{e}_k, \hat{f}_1, \ldots, \hat{f}_k \rangle$ and $E = K \langle \hat{z}, \hat{e}_1, \ldots, \hat{e}_k \rangle$. B is nondegenerate, since this is true over L. It follows that we can change the f_i by elements from E so that they satisfy $[\hat{f}_i, \hat{f}_j] = 0$ for $i, j = 1, \ldots, k$. We then change the $\hat{e}_i$ so that (**) is satisfied. Note that a simple computation using the Jacobi identity shows that the matrix (s_{ij}) defined by

$$[\hat{e}_i, \hat{f}_j] = \hat{s}_{ij} \hat{z}$$

is symmetric. ∎

Proof of Theorems. 1.10, 1.11, 1.14, 1.17. Let $\Gamma \leqslant E(n, 1)$ be a subgroup that acts discontinuously and quasitransitively on affine space. Let H be its kristallographic hull (Theorem 1.5). $H \leqslant E(n, 1)$ acts simply transitively on affine space and $\Delta = H \cap \Gamma$ is of finite index in Γ. H can be conjugated to one of the types of groups described in sections 3 and 5.

A) H is unipotent

By section 3 H is conjugate to $G(\psi, w)$ for suitable ψ and w. The Lie algebra $\mathscr{M}_{\mathbb{Q}}(\Delta)$ is isomorphic over $\mathbb{Q}$ to some Lie algebra $\mathfrak{g}_{\mathbb{Q}}(k, m, \varphi, S)$. It is a simple matter to see that Δ has to be nilpotent of nilpotency class $\leqslant 3$

and has to satisfy (ii) of Theorem 1.1. In Theorem 1.14 we have for every isomorphism class of $\mathfrak{g}_{\mathbb{Q}}(k, m, \varphi, s)$ constructed a particularly nice group $\Gamma_i(n+1, k, m)$ that acts discontinuously and quasitransitively on affine space and satisfies $\mathcal{M}_{\mathbb{Q}}(\Gamma_i(n+1, k, m)) \cong \mathfrak{g}_{\mathbb{Q}}(k, m, \varphi, s)$. It is clear from the results of this section that Theorem 1.14 is valid.

B) H satisfies $\Gamma(H) \leqslant D(n,1)$

We conjugate H so that $H = D_1(d, \eta)$ or $D_2(d_1, \mu, \tau)$. The group of all translations $\leqslant E(n, 1)$ is the unipotent radical of H. By [2] $\lambda(\Delta)$ is discrete in $a(n, 1)$ or $\mathbb{R}^* \times (n, 1)$. In the first case Δ is virtually abelian. In the second the image of $\lambda(\Delta)$ in $\mathbb{R}^*$ has to be discrete and hence cyclic. So in this case Δ is virtually abelian by cyclic. This proves Theorem 1.10.

In case Δ is not virtually abelian take a subgroup $\Delta_0 \leqslant \Delta$ of finite index with $\lambda(\Delta_0) \cap (n, 1) = \langle 1 \rangle$. Δ_0 has $\Delta_0 \cap \mathcal{T}$ as abelian normal subgroup. It is clear that a generator of the cyclic group $\Delta_0/\Delta_0 \cap \mathcal{T}$ acts by a Lorentz type matrix on the discrete group $\Delta_0 \cap \mathcal{T} \leqslant \mathcal{T}$. This proves Theorem 1.15 (i). It is obvious that every group of the type described in (ii) is a lattice in a simply transitive group $H \leqslant E(n, 1)$ with $\lambda(H) \leqslant D(n, 1)$.

C) H satisfies $\lambda(H) \leqslant \hat{P}(n, 1)$

In this case [2] implies that Γ is virtually nilpotent. ■

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